COVER PAGE

Real Analysis Qualifying Examination

Thursday, January 3, 2019

2:30pm – 4:30pm

C-304 Wells Hall

Your Sign-Up Number: ________________________

Note: Attach this cover page to the paperwork you are submitting to be graded. **This number should be the only identification appearing on all of your paperwork – DO NOT WRITE YOUR NAME on any of the paperwork you are submitting.**

Qualifying Exam Real Analysis January 2019

(1) Let f be a complex measurable function on X, μ a positive measure such that $\mu(X) = 1$. Assume $\| f \|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0^+} \| f \|_p = \exp\left\{\int_X \log|f|\right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

(2) It is well-known (Markov's inequality -apparently I was wrong in class when I called it Chebychev's inequality-) that if $f : (X, \mu) \to [0, \infty]$ is in $L^1(\mu)$, for a positive measure μ, then for all $t > 0$, the function

$$\alpha(t) = t \cdot \mu\left(\{x \in X : f(x) > t\}\right)$$

satisfies $\alpha(t) \leq \| f \|_{L^1(\mu)}$. Prove that, if $t \to 0$ or $t \to \infty$, then actually $\alpha(t) \to 0$. (Hint: think of $X = \mathbb{R}$, μ as Lebesgue measure, and think of the set $\{(x, y) : 0 < y < f(x)\}$ as a subset of $\mathbb{R}^2$ and draw pictures...)

(3) Construct (or describe **very** convincingly how to construct) a function $F : \mathbb{R} \to \mathbb{R}$ that is **strictly** increasing (i.e. $x > y \Rightarrow F(x) > F(y)$) but satisfies $F' = 0$ a.e. with respect to Lebesgue measure. You can use any of the "standard" constructions without having to prove their "standard" properties.

(Hint: you can assume Fubini's theorem that if $\{F_j\}$ is a sequence of nondecreasing functions on $[a, b]$ such that $F(x) = \sum_{j=1}^{\infty} F_j(x) < \infty$ for all $x \in [a, b]$, then $F'(x) = \sum_{j=1}^{\infty} F_j'(x)$ for Lebesgue a.e. $x \in [a, b]$.)

(4) Let μ be a regular complex Borel measure on $[0, 1]$.

(a) Define the total variation measure of μ, denoted by $|\mu|$. Note that (and this is **not** part of the problem you are asked to solve) the quantity $\| \mu \|_{TV} = |\mu|([0, 1])$ actually defines a norm on the space of regular complex Borel measures on $[0, 1]$, and you can use this fact throughout the problem.

(b) At the same time, μ is a bounded linear functional over the space $(C([0, 1]), \| \cdot \|_\infty)$. Here $\| \cdot \|_\infty$ denotes the supremum norm $\| f \|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$ (pointwise, for all points).

Define its norm over such space, denoted by $\| \mu \|_{C([0,1])^*}$. Note that (and this is **not** part of the problem you are asked to solve), since the target space for the bounded linear functionals is $\mathbb{C}$ which is complete, then the dual space $C([0, 1])^*$ is complete with the norm $\| \cdot \|_{C([0,1])^*}$.

(c) Prove that $\| \mu \|_{TV} = \| \mu \|_{C([0,1])^*}$.

(d) Consider now μ and ν_n regular complex Borel measures on $[0,1]$, for $n = 1, 2, \ldots$. Assume that $\nu_n \perp \mu$ for all n, and that $\nu_n \to \nu$ in the norm (any of the two norms in the previous parts of the problem). Prove that $\nu \perp \mu$. (Hint: think about (b), Cauchy sequences, and exploit the equality in part (c).)

(5) Let X be a locally compact Hausdorff space, and μ a Borel regular measure on X such that $\mu(\{x\}) = 0$ for all $x \in X$. Prove that there are no atoms of finite measure, i.e. there are no measurable sets A with the property that $0 < \mu(A) < \infty$ and if $B \subset A$ is measurable, then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.